MATLAB Examples

Numerical Integration

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Integration

The integral of a function $f(x)$ is denoted as:

$$\int_{a}^{b} f(x) dx$$
Integration

Given the function:

\[ y = x^2 \]

We know that the exact solution is:

\[ \int_0^a x^2 \, dx = \frac{a^3}{3} \]

The integral from 0 to 1 is:

\[ \int_0^1 x^2 \, dx = \frac{1}{3} \approx 0.3333 \]
An integral can be seen as the area under a curve. Given $y = f(x)$ the approximation of the Area ($A$) under the curve can be found dividing the area up into rectangles and then summing the contribution from all the rectangles (trapezoid rule):

$$A = \sum_{i=1}^{n-1} (x_{i+1} - x_i) \cdot (y_{i+1} + y_i)/2$$
Example:

**Numerical Integration**

We know that the exact solution is:

\[ y(x) = x^2 \quad \rightarrow \quad \int_a^b y(x) \, dx = ? \quad \rightarrow \quad \int_0^1 x^2 \, dx = \frac{a^3}{3} \]

\[ \int_0^1 x^2 \, dx = \frac{1}{3} \approx 0.3333 \]

We use MATLAB (trapezoid rule):

```matlab
x=0:0.1:1; y=x.^2; plot(x,y)

% Calculate the Integral:
avg_y=y(1:length(x)-1)+diff(y)/2;
A=sum(diff(x).*avg_y)
```

\[ A = 0.3350 \]

Students: Try this example
Example:

**Numerical Integration**

We know that the exact solution is:

\[ y(x) = x^2 \]
\[ \int_a^b y(x) \, dx = \frac{a^3}{3} \]

In MATLAB we have several built-in functions we can use for numerical integration:

```matlab
% Calculate the Integral (Trapezoid method):
avg_y = y(1:length(x)-1) + diff(y)/2;
A = sum(diff(x).*avg_y);

% Calculate the Integral (Simpson method):
A = quad('x.^2', 0,1);

% Calculate the Integral (Lobatto method):
A = quadl('x.^2', 0,1);
```
Given the following equation:

\[ y = x^3 + 2x^2 - x + 3 \]

- We will find the integral of \( y \) with respect to \( x \), evaluated from \(-1\) to \(1\)
- We will use the built-in MATLAB functions \texttt{diff()}, \texttt{quad()} and \texttt{quadl()}
Numerical Integration – Exact Solution

The exact solution is:

\[ I = \int_a^b (x^3 + 2x^2 - x + 3) \, dx = \left( \frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} + 3x \right) \bigg|_a^b \]

\[ = \frac{1}{4} (b^4 - a^4) + \frac{2}{3} (b^3 - a^3) - \frac{1}{2} (b^2 - a^2) + 3(b - a) \]

\[ a = -1 \text{ and } b = 1 \text{ gives:} \]

\[ I = \frac{1}{4} (1 - 1) + \frac{2}{3} (1 + 1) - \frac{1}{2} (1 - 1) + 3(1 + 1) = \frac{22}{3} \]
Symbolic Math Toolbox

We start by finding the Integral using the Symbolic Math Toolbox:

```
clear, clc

syms f(x)
syms x

f(x) = x^3 + 2*x^2 -x +3

I = int(f)
```

This gives: \( I(x) = \frac{x^4}{4} + \frac{(2*x^3)}{3} - \frac{x^2}{2} + 3*x \)

Symbolic Math Toolbox

The Integral from a to b:

```matlab
clear, clc
syms f(x)
syms x
f(x) = x^3 + 2*x^2 -x +3
a = -1;
b = 1;
Iab = int(f, a, b)
```

This gives: $I_{ab} = \frac{22}{3} \approx 7.33$

clear, clc

x = -1:0.1:1;
y = myfunc(x);

plot(x,y)

% Exact Solution
a = -1;
b = 1;
Iab = 1/4*(b^4-a^4)+2/3*(b^3-a^3)-1/2*(b^2-a^2)+3*(b-a)

% Method 1
avg_y = y(1:length(x)-1) + diff(y)/2;
A1 = sum(diff(x).*avg_y)

% Method 2
A2 = quad(@myfunc, -1,1)

% Method 3
A3 = quadl(@myfunc, -1,1)

MATLAB gives the following results:

Iab = 7.3333
A1 = 7.3400
A2 = 7.3333
A3 = 7.3333
Integration on Polynomials

Given the following equation:

\[ y = x^3 + 2x^2 - x + 3 \]

Which is also a polynomial. A polynomial can be written on the following general form: 
\[ y(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \]

- We will find the integral of \( y \) with respect to \( x \), evaluated from -1 to 1
- We will use the \texttt{polyint()} function in MATLAB
clear
clc
p = [1 2 -1 3];
polyint(p)

The solution is:
ans =
    0.2500    0.6667  -0.5000    3.0000    0

The solution is a new polynomial:
[0.25, 0.67, -0.5, 3, 0]

Which can be written like this:

\[ 0.25x^4 + 0.67x^3 - 0.5x^2 + 3x \]

We know from an example that the exact solution is:

\[
\int_{a}^{b} (x^3 + 2x^2 - x + 3)dx = \left( \frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} + 3x \right) \bigg|_{a}^{b}
\]

→ So we see the answer is correct (as expected).
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